Cauchy Theorem: more general form

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<u>Def.</u> Let γ be a cycle, $\gamma = \alpha_1\gamma_1 + \dots + \alpha_n\gamma_n$, $z \notin \bigcup_{j=1}^{n}\gamma_j$

The <u>winding number</u> or <u>index</u> of the cycle γ:

$$n(\gamma, z) := \sum_{j=1}^{n} \alpha_j \; n(\gamma_j, z) = \sum_{j=1}^{n} \frac{\alpha_j}{2\pi i} \oint_{\gamma_j} \frac{dw}{w-z}$$

<u>Remark.</u> $P\,dx + Q\,dy$ is exact in Ω if and only if for any cycle γ: $\oint_\gamma P\,dx + Q\,dy = 0$

In complex terms: f has antiderivative iff $\forall$ cycle γ $\oint_\gamma f(z)\,dz = 0$.

<u>Def.</u> A region $\Omega \subset \mathbb{C}$ is called <u>simply-connected</u> if $\hat{\mathbb{C}} \setminus \Omega$ - connected.

<u>Remark</u> (Important!) $\hat{\mathbb{C}}$, not $\mathbb{C}$!

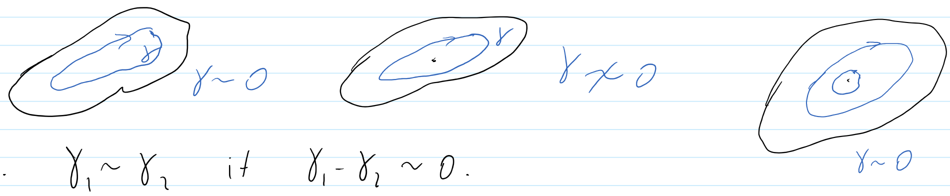
$\Omega = \mathbb{C} \setminus \{0\}$ $\quad \mathbb{C} \setminus \Omega = \{0\}$ - connected $\quad \hat{\mathbb{C}} \setminus \Omega = \{0, \infty\}$ - not connected.

<u>Def</u> Let Ω be a region, $\gamma \subset \Omega$ - a chain.

We say that γ is <u>homologous to 0 with respect to Ω</u> if for any $z \notin \Omega$, $n(\gamma, z) = 0$.

<u>Notation:</u> $\gamma \sim 0$.

Heuristically: γ does not wind around points outside of Ω.



<u>Def.</u> $\gamma_1 \sim \gamma_2$ if $\gamma_1 - \gamma_2 \sim 0$.

<u>Observation.</u> If Ω is simply connected then for <u>any</u> cycle $\gamma \subset \Omega$, $\gamma \sim 0$.

<u>Proof.</u> Let $z \notin \Omega$. Then, since $\hat{\mathbb{C}} \setminus \Omega$ is connected, it belongs to the unbounded component of $\mathbb{C} \setminus \Omega$ (the only one), which is subset of

unbounded component of $\mathbb{C}\setminus\gamma$. So $n(\gamma,z)=0$ ∎

<u>Remark</u>. As proved in Ahlfors: opposite is also true:
$(\forall \gamma \subset \Omega$ - cycle, $\gamma \sim 0) \Rightarrow \Omega$ is simply connected.

<u>Theorem</u> (General Cauchy Theorem).
Let $f \in A(\Omega)$, $\gamma \sim 0$ wrt Ω.
Then $\oint_\gamma f(z)\,dz = 0$

<u>Corollary</u>. $f \in A(\Omega)$, $\gamma_1 \sim \gamma_2 \Rightarrow \oint_{\gamma_1} f(z)\,dz = \oint_{\gamma_2} f(z)\,dz$

We will prove a global version of Cauchy Integral Formula:

<u>Theorem</u> (General Cauchy Integral Formula)
Let $f \in A(\Omega)$, $\gamma \sim 0$ wrt. Ω
Then $\forall z \notin \gamma$,

$$n(\gamma,z)f(z) = \frac{1}{2\pi i}\oint_\gamma \frac{f(\zeta)}{\zeta - z}\,d\zeta.$$

<u>Proof that CIF $\Rightarrow$ Cauchy</u>
Consider $F(z) := f(z)(z-z_0)$ for some $z_0 \in \Omega\setminus\gamma$.
Then $\frac{1}{2\pi i}\oint_\gamma f(z)\,dz = \frac{1}{2\pi i}\oint_\gamma \frac{F(z)}{z-z_0}\,dz = n(\gamma,z_0)F(z_0) = 0$ ∎

<u>Proof of CIF:</u>
<u>Lemma</u> Let $g \in A(\Omega)$, $z_0 \in \Omega$.
Then $\lim_{\substack{(z,\zeta)\to(z_0,z_0)\\ z\neq\zeta}} \frac{g(z)-g(\zeta)}{z-\zeta} = g'(z_0)$

<u>Proof</u>. Need: $\forall \varepsilon>0\ \exists \delta>0 \cdot \sqrt{|z-z_0|^2+|\zeta-z_0|^2}<\delta \Rightarrow \left|\frac{g(z)-g(\zeta)}{\ldots}\right.$

$(z,\zeta)\to(z_0,z_0)$, $z\neq\zeta$

Proof. Need: $\forall \varepsilon>0\ \exists \delta>0: \sqrt{|z-z_0|^2+|\zeta-z_0|^2}<\delta \Rightarrow \left|\frac{g(z)-g(\zeta)}{z-\zeta}-g'(z_0)\right|<\varepsilon$.

Know: $\exists\ \delta>0: |w-z_0|<\delta \Rightarrow |g'(w)-g'(z_0)|<\varepsilon$.

Let $\gamma=[\zeta,z]$ - the interval from ζ to z.

Then
$$g'(z_0)=\oint_\gamma \frac{g'(z_0)}{z-\zeta}\,dw \quad \left(\oint_\gamma dw = z-\zeta\right)$$
$$\frac{g(z)-g(\zeta)}{z-\zeta}=\oint_\gamma \frac{g'(w)}{z-\zeta}\,dw$$

So if $\begin{matrix}|z-z_0|<\delta\\ |\zeta-z_0|<\delta\end{matrix}$ then $\forall w\in\gamma$, $|w-z_0|<\delta \Rightarrow |g'(w)-g'(z_0)|<\varepsilon$.

So $\left|\frac{g(z)-g(\zeta)}{z-\zeta}-g'(z_0)\right|=\left|\oint_\gamma \frac{g'(w)-g'(z_0)}{z-\zeta}\,dw\right|<\frac{\varepsilon}{|z-\zeta|}\cdot \ell(\gamma)=\varepsilon.$

Let now $F(z,\zeta):=\begin{cases}\frac{f(z)-f(\zeta)}{z-\zeta}, & z\neq\zeta\\ f'(z), & z=\zeta.\end{cases}$

Observe: 1) F is a continuous function.

Indeed: (z,ζ) $z\neq\zeta$ - continuity at (z,ζ) obvious.

(z_0,z_0): By Lemma, $\lim\limits_{\substack{(z,\zeta)\to(z_0,z_0)\\ z\neq\zeta}} F(z,\zeta)=f'(z_0)=F(z_0,z_0)$

$\lim\limits_{\substack{(z,\zeta)\to(z_0,z_0)\\ z=\zeta}} F(z,\zeta)=\lim\limits_{z\to z_0} f'(z)=f'(z_0)$, continuity of derivative!

2) $F(z,\zeta)=F(\zeta,z)$

3) For each ζ_0, $z\to F(z,\zeta_0)$ is analytic in Ω.

Indeed $F(z,\zeta_0)=\frac{f(z)-f(\zeta_0)}{z-\zeta_0}$ is analytic for $z\neq\zeta_0$.

$\lim\limits_{z\to\zeta_0} F(z,\zeta_0)(z-\zeta_0)=\lim\limits_{z\to\zeta_0}(f(z)-f(\zeta_0))=0.$

So the singularity at ζ_0 is removable, and $F(\cdot,\zeta_0)$ is analytic in Ω.

Define now: $\Omega'=\{z\in\mathbb{C}\setminus\gamma : n(\gamma,z)=0\}$.

$\boxed{\mathbb{C}\setminus\Omega\subset\Omega'}$ (Because $\gamma\sim 0$).

Define
$$h(z)=\begin{cases}\frac{1}{2\pi i}\oint_\gamma F(z,\zeta)\,d\zeta, & z\in\Omega\\ \frac{1}{2\pi i}\oint_\gamma \frac{f(\zeta)}{\zeta-z}\,d\zeta, & z\in\Omega'.\end{cases}$$

For $z\in\Omega'\cap\Omega$, $\frac{1}{2\pi i}\oint_\gamma F(z,\zeta)\,d\zeta=\frac{1}{2\pi i}\oint_\gamma\frac{f(\zeta)}{\zeta-z}\,d\zeta-\underbrace{\frac{f(z)}{2\pi i}\oint_\gamma\frac{d\zeta}{\zeta-z}}_{f(z)n(\gamma,z)}=\frac{1}{2\pi i}\oint_\gamma\frac{f(\zeta)}{z-\zeta}\,d\zeta.$

So h is well-defined.

$h\in A(\Omega')$ (it is a Cauchy integral of f).

For $z \in \Omega\setminus\gamma$, $h(z) = \underbrace{\frac{1}{2\pi i}\oint_\gamma \frac{f(\xi)}{\xi - z}\,d\xi}_{\text{Cauchy integral}} - n(\gamma, z)f(z) \in A(\Omega\setminus\gamma)$.

analytic

So $h \in A(\mathbb{C}\setminus\gamma)$.

<u>Claim</u>. $\forall z_0 \in \gamma$, h is analytic at z_0.

<u>Proof (of Claim)</u>.

Consider $B(z_0, r) \subset \Omega$. Let Γ be any closed curve in $B(z_0, r)$.

Then $\oint_\Gamma h(z)\,dz = \frac{1}{2\pi i}\oint_\Gamma\left(\oint_\gamma F(z,\xi)\,d\xi\right)dz \overset{\text{continuous}}{=} \oint_\gamma\left(\oint_\Gamma F(z,\xi)\,dz\right)d\xi$.

But $\forall \xi$ $z \to F(z,\xi) \in A(\Omega) \Rightarrow \oint_\Gamma F(z,\xi)\,dz = 0$.

So $\forall$ Γ - closed, $\Gamma \subset B(z_0, r)$ we have $\oint_\Gamma h(z)\,dz \equiv 0$.

So, by Morera, $h \in A(B(z_0, r))$ ■

Thus $h \in A(\mathbb{C})$.

Also $\lim_{|z|\to\infty} h(z) = \lim_{|z|\to\infty}\left|\int \frac{f(\xi)}{z - \xi}\,d\xi\right| = 0$.

So, by Maximum principle, $h \equiv 0$

So, for $z \in \Omega$

$$0 = h(z) = \frac{1}{2\pi i}\oint_\gamma \frac{f(\xi)}{\xi - z}\,d\xi - n(\gamma, z) f(z) \;■$$

<u>Corollary</u>. If Ω is simply connected,

then $\forall f \in A(\Omega)$, $\gamma \subset \Omega$ - cycle,

1) $\oint_\gamma f(z)\,dz = 0$

2) $f(z_0)n(\gamma, z_0) = \frac{1}{2\pi i}\oint_\gamma \frac{f(z)}{z - z_0}\,dz$, if $z_0 \in \Omega\setminus\gamma$.

<u>Proof</u>. $\gamma \sim 0$ ■

<u>Corollary</u> Let Ω be simply connected. $f \in A(\Omega)$. $\forall z \in \Omega$ $f(z) \neq 0$.

<u>Then</u> $\exists\ g \in A(\Omega)$: $e^g = f$ (branch of logarithm)

$\forall n \in \mathbb{N}$ $\exists h \in A(\Omega)$: $h^n = f$ (branch of n-th root).

<u>Proof</u>. Note that $\frac{f'(z)}{f(z)} \in A(\Omega)$.

Thus $\exists \tilde{g}$: $\tilde{g}'(z) = \frac{f'(z)}{f(z)}$, $\tilde{g} \in A(\Omega)$ (antiderivative).

Fix $z_0 \in \Omega$. Take $g(z) := \tilde{g}(z) - \tilde{g}(z_0) + \operatorname{Log} f(z_0)$

Then 1) $e^{g(z_0)} = e^{\operatorname{Log} f(z_0)} = f(z_0)$

2) $\left(f(z) e^{-g(z)}\right)' = f'(z) e^{-g(z)} - f(z) \cdot \underbrace{g'(z)}_{\frac{f'(z)}{f(z)}} e^{-g(z)} = 0$

Thus $f(z) e^{-g(z)} \equiv const = f(z_0) e^{-g(z_0)} = 1 \Rightarrow$

$f(z) = e^{g(z)}$

Now take $h(z) := \exp\left(\frac{g(z)}{n}\right)$

<u>Corollary</u> If γ bounds Ω and $f \in A(\Omega \cup \gamma)$, then

$$\frac{1}{2\pi i} \oint_\gamma \frac{f(z)}{z - z_0} dz = \begin{cases} f(z_0), & z_0 \in \Omega \\ 0, & z_0 \notin \Omega \\ & z_0 \notin \gamma \end{cases}$$